Midsemestral examination 2007
M.Math. IInd year
Algebraic Number Theory : B.Sury

Instructions :
(i) *This is an open-book examination where any one book (other than a problem book) is allowed for consultation.*
(ii) *Answer any 3 questions from the global part G1 to G5 and any 2 from the local part L1 to L5.*
(iii) *By a p-adic field, we mean a finite extension of* $\mathbf{Q}_p$. *The notation* ζ_n *is used for any primitive* n*-th root of unity.*

G 1.
Show that a Dedekind domain with only finitely many prime ideals must be a PID.

OR

Show that any ideal $I \neq 0$ in a Dedekind domain can be generated by two elements, one of which can be an arbitrary non-zero element.

G 2.
For any n, and any prime number $p \equiv 1 \bmod n$, describe the prime ideals of $\mathbf{Z}[\zeta_n]$ lying over p.

OR

Prove the 'cyclotomic reciprocity law' : a prime p splits completely in $\mathbf{Q}[\zeta_n]$ if and only if $p \equiv 1 \bmod n$.

G 3.
Let K denote the unique subfield of $\mathbf{Q}[\zeta_{31}]$ of degree 6 over $\mathbf{Q}$. Prove that $\mathcal{O}_K$ cannot be of the form $\mathbf{Z}[\alpha]$ for any α.

G 4.
Let K be a cubic extension of $\mathbf{Q}$ whose discriminant is -49. Prove that the class number of K is 1.

OR

Prove that the class number of $\mathbf{Q}(\sqrt{10})$ is 2.

G 5.
Find (with proof) the fundamental unit of $\mathbf{Q}(\sqrt{13})$.

L 1.
Let k be a complete field with respect to a discrete valuation. Let $\mathcal{O}_k$ and $\mathcal{M}$ denote the valuation ring and is maximal ideal. Prove that $\mathcal{O}_k$ is compact if and only if $\mathcal{O}_k/\mathcal{M}$ is a finite field.

OR

Use Hensel's lemma to find the structure of the group $\mathbf{Z}_2^*/(\mathbf{Z}_2^*)^2$.

L 2.
Prove that $\mathbf{Q}_p$ and $\mathbf{Q}_q$ are non-isomorphic if $p \neq q$.

OR

Find the radius of convergence of the exponential series $exp(x)$ over $\mathbf{Q}_p$.

L 3.
Let K denote an algebraic closure of $\mathbf{Q}_p$ and $\mathcal{O}$ denote the integral closure of $\mathbf{Z}_p$ in K. Prove that $\mathcal{O}$ is not Noetherian.
Hint : Show that $\mathcal{O}$ has only one nonzero prime ideal and it is not finitely generated.

OR

Let K denote an algebraic closure of $\mathbf{Q}_p$. Let $\{b_n\}$ be a sequence of roots of unity of order coprime to p in K satisfying the following properties :
$b_1 = 1, b_n \in \mathbf{Q}_p(b_{n+1}), [\mathbf{Q}_p(b_{n+1}) : \mathbf{Q}_p(b_n)] > n.$
Deduce that K is not complete by showing that the series $\sum_n b_n p^n$ does not converge in K.

L 4.
Let k be a finite extension of $\mathbf{Q}_p$. Use Hensel's lemma to describe the unique unramified extension of given degree n over k.

OR

Prove that a totally ramified extension of a p-adic field k is given by adjoining the root of an Eisenstein polynomial over k.

L 5.
Let k be a p-adic field (that is, a finite extension of $\mathbf{Q}_p$). Write $\mathcal{O}$ and P for its ring of integers and the maximal ideal. If $p\mathcal{O} = P^e$, prove that e must be divisible by $p-1$ in order that k contains a primitive p-th root of unity.